WS 2016/17

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Symplectic Geometry

Homework 7

Exercise 1. (10 points)
Consider a trivial bundle $E = X \times \mathbb{R}^m \xrightarrow{\pi} X$ over an n-dimensional manifold X. Let N_E be the fiber conormal bundle, i.e. the space of points (e, η) of T^*E such that η vanishes on the kernel of $d\pi|_e$. As shown in the lecture, in this case $N_E = T^*X \times \mathbb{R}^m \times \{0\} \subset T^*X \times T^*(\mathbb{R}^m) = T^*(X \times \mathbb{R}^m)$.

(a) Prove that N_E is a coisotropic submanifold of T^*E (equipped with the canonical symplectic structure of a cotangent bundle).

Recall the coisotropic reduction of symplectic vector spaces. Note that the derivative (push forward) of the natural projection $pr\colon N_E \to T^*X$, at $(e, \eta) \in N_E$, is the coisotropic reduction map $T_{(e,\eta)}N_E \to T_{(e,\eta)}N_E/(T_{(e,\eta)}N_E)^\omega$. Therefore T^*X is called a coisotropic reduction of T^*E with respect to N_E.

(b) Given $f\colon X \times \mathbb{R}^m \to \mathbb{R}$, what is the coisotropic reduction of the graph of df with respect to N_E, i.e. what is $pr(N_E \cap (\operatorname{graph} df))$?

Exercise 2. (10 points)
Prove that if $L_1 \subset T^*B_1$ has generating function $F_1 : B_1 \times \mathbb{R}^{N_1} \to \mathbb{R}$ and $L_2 \subset T^*B_2$ has generating function $F_2 : B_2 \times \mathbb{R}^{N_2} \to \mathbb{R}$ then $L_1 \times L_2 \subset T^*B_1 \times T^*B_2 \equiv T^*(B_1 \times B_2)$ has generating function $F_1 \oplus F_2 : (B_1 \times B_2) \times (\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \to \mathbb{R}$.
(The notation $F_1 \oplus F_2$ means that $F_1 \oplus F_2(b_1, v_1, b_2, v_2) = F_1(b_1, v_1) + F_2(b_2, v_2)$. The base variable is $(b_1, b_2) \in B_1 \times B_2$, and the fiber is $(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$.)

Exercise 3. (10+10 points)
Exercises 1 and 3 from Homework 6 (on page 50) in *Lectures on Symplectic Geometry* by A. Cannas da Silva. Also available online at: http://www.mi.uni-koeln.de/~pabiniak/sg.html

Bonus Exercise. (10 points)
Suppose that $L \subset T^*B$ has generating function $F\colon B \times \mathbb{R}^N \to \mathbb{R}$, and $A_h : T^*B \to T^*B$ is a symplectomorphism of the form $A_h(b, v) = \big(b, v\big) + (b, dh_{|b})$ for some function $h : B \to \mathbb{R}$. Prove that the function $F + h\colon B \times \mathbb{R}^N \to \mathbb{R}$, defined by $(F + h)(b, v) = F(b, v) + h(b)$, is a generating function for $A_h(L) \subset T^*B$.

Bonus Exercise. (10 points)
Prove that if a Lagrangian submanifold L of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2m})$ has generating function $F : (\mathbb{R}^{2n} \times \mathbb{R}^{2m}) \times \mathbb{R}^N \to \mathbb{R}$ then the reduction $\overline{L} \subset T^*\mathbb{R}^{2n}$ of L with respect to the coisotropic submanifold $V = \mathbb{R}^{2n} \times \mathbb{R}^{2m} \times (\mathbb{R}^{2n})^* \times \{0\}$ of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2m})$ has a generating function $\overline{F} : \mathbb{R}^{2n} \times (\mathbb{R}^{2m} \times \mathbb{R}^N) \to \mathbb{R}$, $\overline{F}(u; v, \zeta) = F(u, v; \zeta)$. (Note that the only difference between F and $\overline{F}$ is in the splitting of the domain into base and fiber. Parentheses and signs ; separate base variable from fiber variable.)

Hand in: Thursday December 8th
in the exercise session
in Übungsraum 1, MI